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Single Polygon Counting on Cayley Tree of Order 3

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Abstract We showed that one form of generalized Catalan numbers is the solution to the problem of finding different connected component with finite vertices containing a fixed root for the semi-infinite Cayley tree of order 3. We give the formula for the full graph, Cayley tree of order 3 which is derived from the generalized Catalan numbers. Using ratios of Gamma functions, two upper bounds are given for problem defined on semi-infinite Cayley tree of order 3 as well as the full graph.

Keywords Cayley tree · Generalized Catalan numbers · Contour method

1 Introduction

Many real world problems can be translated into the language of networks: information travel in the cyber space or people moving from one location to another. The basic entities of a system are represented by a set of nodes and they are connected by edges due to some relationship established between the nodes. As was argued in [1], network theory even provide an explanatory framework of interrelationships of how countries as tourism destinations interact, relate and evolve.

Some connected networks are trees i.e. graphs with no loops and exist only as single shortest path from one node to another. The statistical characterization of real networks displays a large number of node degrees, denoted by n [2]. These networks always grow in a power-law behavior $C_n \sim n^{-m}$, where $m > 1$. Power-law behaviors are often found in social and economical phenomena such as population distribution, subway system [3], and etc.

On the other side, mathematicians are interested in finding very sharp upper and lower bounds for certain functions e.g. gamma functions and ratios of gamma functions [4–6], which is also in the form of power-law. In this paper, we will employ some of these established results to generalize our previous result [7] where we have shown that the problem

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of finding the number of different connected component with exactly n vertices in a semi-infinite Cayley tree of order 2 (for details of Cayley tree, one can refer [8]) is exactly the ordinary Catalan numbers [9]:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

There are many generalizations of the Catalan numbers [10–12]. In this paper we are going to consider generalized Catalan numbers [11] of the form

$$C_{n,k} = \frac{1}{(k-1)n+1}\binom{kn}{n},$$

with $k = 3$. The generalized Catalan numbers have different interpretations e.g. Dyck paths [13] and non-crossing trees in a circle [14].

In [7], we extended the investigation from the semi-infinite tree to the complete graph of the Cayley tree of order 2 by another less known sequence. The relationship of these two sequences is linked by another aspect, other than some of the connections previously established. The motivation of finding such an estimate of these numbers is to solve a combinatorial problem in the contour method [15–17] for lattice models. Despite the fact that the properties of Catalan numbers were extensively studied [9, 11], we restrict ourself only to the problem of finding a suitable estimate such that, the Catalan numbers satisfy

$$C_n < a^n/n$$

where $a \in \mathbb{R}^+$, $n \in \mathbb{N}$ and a is to be determined as the center of our problem.

In the integer lattice $\mathbb{Z}^2$, it is well known that the number of different non-equivalent polygon of contour length n is less than $4 \cdot 3^{n-1}$ (the number of all polygonal paths passing through a given point), which could be found in all standard texts, e.g. [18] discussing phase transitions on $\mathbb{Z}^2$. For the Gibbs measures constructed in the Ising model on $\mathbb{Z}^2$ (for detail see [18, 19]), the probability of a constant i-configuration, $\Lambda \subset \mathbb{Z}^2$, is then

$$P^{i=+1}_{\Lambda,\beta}(V_-) < \sum_{n=4,6,\ldots}^{\infty} 4 \cdot 3^n \exp(-2\beta n), \tag{1}$$

where n is the contour length, β is the inverse temperature and V_- is the event that the center has a negative spin. This estimate is required in the proof of existence of phase transition, i.e. the probability of the event on the $\mathbb{Z}^2$ which is directly related to this estimate approaches zero when $n \to \infty$. This non-symmetric result shows that the spin at a fixed vertice is influenced by the boundary condition chosen, which gives rise to the phenomenon of phase transition. The advantage of the contour method is that no explicit calculation of the probability is required. A more comprehensive and alternative version of the contour method, which is now known as Pirogov-Sinai theory, was given by Zahradnik [20].

Although, Catalan numbers only correspond to the problem of all possible finite subgraph of semi infinite Cayley tree, we are also interested in another sequence which is associated to the finite subgraph of the full graph. In this paper, we would like to extend our study from Cayley tree of order 2 to order 3, and hope to generalize to order k in future. We first find an expression for the sequence for the counting problem on semi-infinite Cayley tree of order 3, and then express these Catalan numbers in terms of ratio of gamma functions. Then we solve the same problem (all possible finite sub-graph) for the complete graph of Cayley tree of order 3.

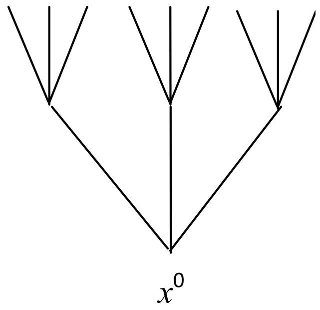

Fig. 1 A semi infinite Cayley tree of order 3 with root x^0, Γ^3_{semi}

Definition 1 *A semi infinite Cayley tree of order* 3, *denoted as* Γ^3_{semi}, *is a graph with no cycles*, *each vertex emanates* 4 *edges except the root denoted as* x^0 *which emanates only* 3 *edges* (*see Fig*. 1).

We denote the set of all vertices as V and the set of all edges as L, i.e. $\Gamma^3_{\text{semi}} = (V, L)$.

2 Semi-infinite Cayley Tree of Order 3, Γ^3_{semi}

In this section, we are going to establish an asymptotic behavior of the number of n vertices connected component containing a fixed root x^0 on the semi-infinite Cayley tree of order 3.

Let Γ^3_{semi} be the semi infinite Cayley tree of order 3 and by C_n we denote the number of different connected components with exactly n vertices containing a root $x^0 \in V$, vertices of $\Gamma^3_{\text{semi}} = (V, L)$. Similar as in [7], we adopt the phrase "single polygon" from the same problem but defined on integer lattice $\mathbb{Z}^d$ (for details see [18]). In this counting problem, we only count the connected component with finite vertices. Note that this problem is the same as constructing a connected labeled ternary tree with n vertices [14], but slightly different (see Fig. 2) with those counting doesn't consider the coordination of the successors of a vertex (non-labeled).

The main result is the recursion of the C_n, i.e.

Theorem 1 *C_n can be written in nonlinear recursion as*:

$$C_n = \sum_{i+j+k=n-1} C_i C_j C_k, \quad C_0 = 1 \tag{2}$$

where $i \geq 0, j \geq 0, k \geq 0, i, j, k \in \mathbb{Z}$.

Proof We divide the problem of finding n number of vertices which containing a root x^0 into 3 parts consist of i, j and k number of vertices connect to the root, i.e. the successors of the root. The total combination is the product of the number of all successors $C_i C_j C_k$. Then C_n is sum over $C_i C_j C_k$ for all $i + j + k = n - 1$. We define $C_0 = 1$ which is simply a result from observation. □

Equation (2) can also be written as

$$C_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} C_i C_j C_{n-i-j-1} \tag{3}$$

through replacing k by $n-i-j-1$ and single sum by two sums. The estimate for the problem defined on semi infinite Cayley tree of order 3 is straightforward after we obtain explicit form of C_n as follows:

Corollary 1 *C_n is the generalized Catalan numbers,*

$$C_n = \frac{1}{2n+1}\binom{3n}{n}, \qquad n = 0, 1, 2, \ldots. \tag{4}$$

Proof Using a generating function

$$u = \sum_{i=0}^{\infty} C_i x^i = 1 + x + 3x^2 + 12x^3 + \cdots, \tag{5}$$

we can obtain following relationship using (3)

$$\begin{aligned} u^3 &= \left(\sum_{i=0}^{\infty} C_i x^i\right)\left(\sum_{j=0}^{\infty} C_j x^j\right)\sum_{k=0}^{\infty} C_k x^k \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} C_i C_j C_k x^{i+j+k} \\ &= \sum_{n=0}^{\infty} C_{n+1} x^n. \end{aligned}$$

Multiplying both sides by x,

$$\begin{aligned} xu^3 &= \left(\sum_{n=0}^{\infty} C_{n+1} x^{n+1} + 1\right) - 1, \\ xu^3 &= u - 1. \end{aligned}$$

The unique solution to this equation, by Lagrange inversion formula is

$$C_n = \frac{1}{2n+1}\binom{3n}{n}.$$

Theorem proved. □

Remark Note that the formula C_n (4) is a well known one on ternary tree [5, 14], prove using generating functions and Lagrange inversion formula is also standard way. We give the proof based on the main result i.e. the recursion (2) obtained just for the sake of completeness of the paper.

The first few terms of the sequence are given as (see Sloane, [21], A000108)

$$1, 1, 3, 12, 55, 273, 1428, 7752, 43263, 246675, 1430715, \ldots$$

From result above, we can express C_n in terms of gamma functions.

Using (2) and (4), one could establish following identities:

$$\frac{1}{2n+1}\binom{3n}{n} = \sum_{i+j+k=n-1} \frac{1}{2i+1}\binom{3i}{i}\frac{1}{2j+1}\binom{3j}{j}\frac{1}{2k+1}\binom{3k}{k}$$
$$= \sum_{i=0}^{n-1}\sum_{j=0}^{n-i-1} \frac{1}{2i+1}\binom{3i}{i}\frac{1}{2j+1}\binom{3j}{j}\frac{1}{2(n-i-j)-1}\binom{3(n-i-j-1)}{n-i-j-1}.$$

Corollary 2 *C_n can be expressed as*:

$$C_n = \sqrt{\frac{3}{16\pi}}\left(\frac{27}{4}\right)^n \frac{\Gamma(n+\frac{2}{3})}{\Gamma(n+\frac{3}{2})}\cdot\frac{\Gamma(n+\frac{1}{3})}{\Gamma(n+1)}. \tag{6}$$

Proof It is not difficult to deduce the recursion from theorem above:

$$C_{n+1} = \frac{3\cdot(3n+2)(3n+1)}{(2n+3)(2n+2)}C_n$$
$$= \frac{3\cdot 3\cdot 3\cdot(n+\frac{2}{3})(n+\frac{1}{3})}{2\cdot 2\cdot(n+\frac{3}{2})(n+1)}C_n$$
$$= \left(\frac{27}{4}\right)^{n+1}\frac{(n+\frac{2}{3})(n+\frac{1}{3})}{(n+\frac{3}{2})(n+1)}\frac{(n+\frac{2}{3}-1)(n+\frac{1}{3}-1)}{(n+\frac{3}{2}-1)(n+1-1)}\cdot\cdots\cdot\frac{\frac{2}{3}\cdot\frac{1}{3}}{\frac{3}{2}}C_0.$$

Therefore, one can expand C_n to C_0 and introducing $\frac{\Gamma(2/3)\Gamma(1/3)}{\Gamma(3/2)}$ to obtain gamma function and

$$C_n = \left(\frac{27}{4}\right)^n\sqrt{\frac{3}{16\pi}}\cdot\frac{\Gamma(n+\frac{2}{3})}{\Gamma(n+\frac{3}{2})}\cdot\frac{\Gamma(n+\frac{1}{3})}{\Gamma(n+1)}.$$ □

An estimate for the C_n is obtained as follows:

Corollary 3 *The inequality of C_n is given as*

$$\left(\frac{27}{4}\right)^{n-1}\Big/ n^{3/2} \le C_n < \sqrt{\frac{3}{16\pi}}\left(\frac{27}{4}\right)^n\Big/ n^{3/2} \quad \textit{for } n>0. \tag{7}$$

Proof From an elegant result proven by Wendel [4], i.e.

$$\frac{\Gamma(x+s)}{\Gamma(x)} < x^s,$$

where $x>0$ and s is a real constant such that $0<s<1$, we can show that

$$\frac{\Gamma(n+2/3)}{\Gamma(n+3/2)}\frac{\Gamma(n+1/3)}{\Gamma(n+1)} = \frac{\Gamma(n+1/2+1/6)}{(n+1/2)\Gamma(n+1/2)}\frac{\Gamma(n+1/3)}{n\cdot\Gamma(n)}$$
$$< \frac{(n+1/2)^{1/6}}{(n+1/2)}\cdot\frac{n^{1/3}}{n}$$

$$
\begin{aligned}
&= \frac{1}{(n+1/2)^{5/6}} \cdot \frac{1}{n^{2/3}} \\
&< \frac{1}{n^{5/6}} \cdot \frac{1}{n^{2/3}} \\
&= \frac{1}{n^{3/2}}.
\end{aligned}
$$

From (7) and inequality above, one can show immediately the right hand side of the inequalities

$$
\begin{aligned}
C_n &= \sqrt{\frac{3}{16\pi}} \left(\frac{27}{4}\right)^n \frac{\Gamma(n+\frac{2}{3})}{\Gamma(n+\frac{3}{2})} \frac{\Gamma(n+\frac{1}{3})}{\Gamma(n+1)} \\
&< \sqrt{\frac{3}{16\pi}} \left(\frac{27}{4}\right)^n \frac{1}{n^{3/2}}.
\end{aligned}
$$

The left hand side, we are going to prove using mathematical induction. For $n = 1$, $C_0 = 1$, the equality holds. Suppose that $C_n \geq (\frac{27}{4})^{n-1}/n^{3/2}$ and multiply both side by $\frac{3\cdot(3n+2)(3n+1)}{(2n+3)(2n+2)}$,

$$
\begin{aligned}
\frac{3\cdot(3n+2)(3n+1)}{(2n+3)(2n+2)} C_n &\geq \left(\frac{27}{4}\right)^{n-1} \Big/ n^{3/2} \cdot \frac{3\cdot(3n+2)(3n+1)}{(2n+3)(2n+2)} \\
C_{n+1} &\geq \left(\frac{27}{4}\right)^{n-1} \Big/ n^{3/2} \cdot \frac{3\cdot(3n)(3n+3)}{(2n+3)(2n+2)} \\
&> \left(\frac{27}{4}\right)^{n-1} \Big/ n^{3/2} \cdot \frac{3\cdot 3\cdot(3n)}{2\cdot 2\cdot(n+2)} \\
&> \left(\frac{27}{4}\right)^{n} \Big/ (n+1)^{3/2}.
\end{aligned}
$$

Hence the result is proven. □

From result above and numerical study, C_n is observed to be at order $(\frac{27}{4})^n/n^{3/2}$. Using Stirling's approximation, we can find that

$$
C_n \sim \sqrt{\frac{3}{16\pi}} \left(\frac{27}{4}\right)^n \Big/ n^{3/2}
$$

which is the upper bound of the result. We shall give a rather close and simple (rational number) upper bound for the estimate required in contour method as

$$
\left(\frac{27}{4}\right)^n \Big/ n = \left(\frac{3\cdot 3\cdot 3}{2\cdot 2}\right)^n \Big/ n.
$$

Comparing to order 2, which the estimate is given as

$$
\left(\frac{2\cdot 2}{1}\right)^n \Big/ n.
$$

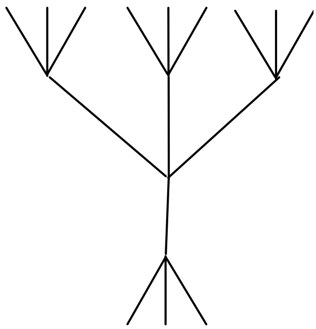

Fig. 2 A Cayley tree of order 3, Γ^3

We conjecture for arbitrary order k the estimate can be written as

$$c_{n,k} < \left(\frac{k \cdot k^{k-1}}{(k-1)^{k-1}}\right)^n \Big/ n$$

and, for sufficiently large k, this term for arbitrary k always less than

$$c_{n,k} < \frac{(k \cdot e)^n}{n}, \quad \forall n, k$$

where $\lim_{k\to\infty}(1+1/k)^k = e$.

3 Complete Graph, Γ^3

In this section, we shall use the result above to derive the inequalities for the similar problem but in the complete graph of Cayley tree of order 3. First, we let D_n be the number of n vertices connected component containing a fixed root x^0, subset to V, vertices of $\Gamma^3 = (V, L)$ that is Cayley tree of order 3 (see Fig. 2).

Theorem 2 *D_n can be written in a nonlinear recursion of C_n as*:

$$D_n = \sum_{r=1}^{n-1} C_r C_{n-r-1} \quad \textit{for } n > 0. \tag{8}$$

Proof We decompose the problem of finding the number of connected component of n number of vertices containing a root x^0 into counting: (i) r number of vertices which containing x^0, i.e. C_r and (ii) $n-r-1$ number of vertices which containing a root y which is another successor of x^0, i.e. C_{n-r-1}. Since the former C_r must always count x^0, it should range from 1 to $n-1$. The total D_n is then the sum of all $C_r C_{n-r-1}$ where r range from 1 to $n-1$. □

This formula lead to the connection between the sequence C_ns' and D_ns'. The first few terms of D_n are listed as follow (one can also see Sloane [21], A006629).

$$1, 4, 18, 88, 455, 2448, 13\,566, 76\,912, 444\,015, 2\,601\,300, \ldots$$

From the same site, we have

$$D_n = \frac{2}{n+1}\binom{3n}{n-1}. \tag{9}$$

Note that the problem that we defined for D_n doesn't allow us to define D_0 where x^0 has to be present all time.

From (8) and (9), we can derive following identity as well:

$$\frac{2}{n+1}\binom{3n}{n-1} = \sum_{r=1}^{n-1} \frac{1}{2r+1}\binom{3r}{r} \cdot \frac{1}{2n-2r-1}\binom{3(n-r-1)}{n-r-1} \quad \text{for } n > 0. \tag{10}$$

Also, from the structure of the tree and same argument, we have following relation

$$D_{n+1} = \sum_{i+j+k+l=n} C_i C_j C_k C_l, \quad C_0 = 1,\ n > 1,$$

where $i, j, k, l \in N$.

Corollary 4 *The estimate of D_n is*

$$D_n < \sqrt{\frac{3}{4\pi}} \frac{n}{n+1} \left(\frac{27}{4}\right)^n \Big/ n^{3/2}. \tag{11}$$

Proof From (9), one can show that

$$D_n = \frac{2n}{n+1} C_n$$

and from inequality $C_n < \sqrt{\frac{3}{16\pi}} (\frac{27}{4})^n / n^{3/2}$, we can prove easily that

$$D_n < \sqrt{\frac{3}{4\pi}} \frac{n}{n+1} \left(\frac{27}{4}\right)^n \Big/ n^{3/2}$$

which completes the proof. □

One can also obtain a simpler form as

$$D_n < \left(\frac{27}{4}\right)^n \Big/ n.$$

For arbitrary order k, we conjecture that

$$D_{n,k} < \frac{(k \cdot e)^n}{n}, \quad \forall n > 0,\ k > 1$$

by similar argument.

Remark In contour method, we are more interested in the inequality for the full graph. In this paper, we first showed that C_n can be expressed in a form of non-linear recursion.

Next, it could be proved that C_n is the well known generalized Catalan number. Then, from this formula we obtained formula for D_n, which correspond to the full graph, rather than the generalized Catalan numbers on ternary tree. The full graph is a homogeneous one, any vertex can be chosen as the fixed root. Further, we are employing inequality of ratio of gamma functions [4] to give two asymptotic estimates for C_n which is more than just a counting problem on C_n. The solution for D_n was obtained using the estimate of C_n. Similar ideas could be applied to other estimate generated by generalized Catalan numbers. A less motivated result are those identities of Catalan number through the recursions (2) and (8) which derived from the topological structure of the trees (Γ^3_{semi} or Γ^3).

Note that Catalan numbers (semi-infinite Cayley tree of order 2) and generalized Catalan numbers (semi-infinite Cayley tree of order 3) both asymptotically grow at order $n^{-3/2}$. We conjecture that other generalized Catalan numbers (for problem defined on semi-infinite Cayley tree of order $k > 3$) also grow in the same manner.

4 Conclusion

We have shown that one form of generalized Catalan numbers is the solution to finding the number of different connected component containing a fix root x_0 on semi-infinite Cayley tree of order 3, i.e. C_n. From this formula, we show that the formula for the full graph, namely D_n, is generated by the same Catalan numbers. Some identities of Catalan numbers can be established from this connection. Then we gave an estimate for both semi-infinite Cayley tree of order 3 (Γ^3_{semi}) and full graph (Cayley tree of order 3, Γ^3). Also we conjecture the estimate for case of arbitrary order k.

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References

1. Miguens, J.I.L., Mendes, J.F.F.: Travel and tourism: Into a complex network. Physica A **387**(12), 1 (2008)
2. Goltsev, A.V., Dorogovtsev, S.N., Mendes, J.F.F.: Critical phenomena in complex networks. Rev. Mod. Phys. **80**, 1275 (2008)
3. Lee, K., Jung, W.S., Park, J.S., Choi, M.Y.: Statistical mechanics and its applications. Physica A **387**(24), 6231–6234 (2000)
4. Wendel, J.G.: Note on the Gamma function. Am. Math. Mon. **55**(9), 563–564 (1948)
5. Gessel, I.M., Xin, G.: The generating function of ternary trees and continued fractions. J. Comb. **13**(2), R53 (2006)
6. Yu, Y.: An inequality for ratios of gamma functions. J. Math. Anal. Appl. **352**(2), 967–970 (2009)
7. Pah, C.H.: An application of Catalan number on Cayley tree of order 2: Single polygon counting. Bull. Malays. Math. Sci. Soc. **31**(2), 175–183 (2008)
8. Baxter, R.J.: Exactly Solved Models in Satistical Mechanics. Academic Press, London (1982)
9. Stanley, R.P.: Enumerative Combinatorics, vol. 2. Cambridge University Press, Cambridge (1999)
10. Klarner, D.A.: Correspondences between plane trees and binary sequences. J. Comb. Theory **9**, 401–411 (1970)
11. Hilton, P., Pedersen, J.: Catalan numbers, their generalization, and their uses. Math. Intell. **13**, 64–75 (1991)
12. Solomon, N., Solomon, S.: A natural extension of Catalan numbers, J. Integer Seq. **11**, Article 08.3.5 (2008)
13. Hilton, P., Pedersen, J.: Catalan numbers, their generalization, and their uses. Math. Intell. **13**(2), 64–75 (1991)

14. Panholzer, A., Prodinger, H.: Bijections for ternary trees and non-crossing trees. Discrete Math. **250**, 181–195 (2002)
15. Griffiths, R.B.: Peierls' proof of spontaneous magnetization in a two-dimensional Ising ferromagnet. Phys. Rev. **136**(2), 437–439 (1964)
16. Pirogov, S.A., Sinai, Ya.G.: Phase diagrams of classical lattice systems, I. Theor. Math. Phys. **25**, 1185–1192 (1975)
17. Pirogov, S.A., Sinai, Ya.G.: Phase diagrams of classical lattice systems, II. Theor. Math. Phys. **26**, 39–49 (1976)
18. Minlos, R.A.: Introduction to Mathematical Statistical Physics. University Lecture Series, vol. 19. American Mathematical Society, Providence (2000)
19. Georgii, H.-O.: Gibbs Measures and Phase Transitions. Walte de Cruyter, Berlin (1998)
20. Zahradnik, M.: A short course on the Pirogov-Sinai theory. Rend. Math. Ser. **VII**, 559–581 (1984)
21. Sloane, N.J.A.: The on-line encyclopedia of integer sequences, published electronically at http://www.research.att.com/~njas/sequences/. Accessed 23 October 2009